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1978 J. Phys. A: Math. Gen. 11 2375

(http://iopscience.iop.org/0305-4470/11/12/005)

J. Phys. A: Math. Gen., Vol. 11, No. 12, 1978. Printed in Great Britain

General relationship between linear and nonlinear realisations of supersymmetry

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Received 1 June 1978

Abstract. The close correspondence between the superfield approach to the supersymmetry and the Volkov–Akulov nonlinear realisation is established. We show that any superfield, by certain field-dependent changes of variables x_μ, θ_α, can be transformed to the form in which it is represented by nonlinearly transforming components. Conversely, there always exist functions of the nonlinear realisation objects which transform according to the linear supertransformation law. We derive general theorems and explicit formulae which describe the transition from the linear realisation to the nonlinear one and vice-versa.

1. Introduction

The supersymmetry can be implemented within two different approaches. One of them is the linear realisation (Golfand and Lichtman 1971, Wess and Zumino 1974) defined, in its most elegant form, on superfields $\phi_k(x, \theta)$ (Salam and Strathdee 1974, 1975)†

$$\Phi'_k(x, \theta) = \Phi_k\left(x + \frac{1}{2\mathrm{i}}\bar{\epsilon}\gamma\theta, \theta - \epsilon\right) \tag{1}$$

ϵ being a constant spinor parameter and k the Lorentz index. The other is the Volkov–Akulov (1972, 1973) nonlinear realisation which involves as a basic entity the nonlinearly and inhomogeneously transforming Goldstone spinor $\lambda(x)$

$$\delta\lambda(x) = \epsilon + \frac{1}{2\mathrm{i}}\bar{\epsilon}\gamma_\mu\lambda(x)\,\partial^\mu\lambda(x) \tag{2}$$

and acts on other, unpreferred fields $\sigma_k(x)$ as follows

$$\delta\sigma_k(x) = \frac{1}{2\mathrm{i}}\bar{\epsilon}\gamma_\mu\lambda(x)\,\partial^\mu\sigma_k(x) \tag{3}$$

(see, also, Pashnev 1974). The nonlinear realisation exhibits most purely the concept of spontaneous supersymmetry breaking.

In the case of usual, Bose symmetries, analogous approaches are known to be related with each other (Coleman, Wess and Zumino (CWZ) 1969) and one may expect something of this kind also in the case of supersymmetry. For instance, in 1974

† We are using the Majorana formalism. Our conventions on metrics and γ-matrices coincide with those of Salam and Strathdee (1975).

Ogievetsky constructed certain linearly transforming functions of the Goldstone spinor $\lambda(x)$ in the space-time of dimensionality $2+1$. The analogous result for a scalar supermultiplet in the Minkowski space–time has been reported by Zumino (1974). However the general structure of the relationship between two realisations of supersymmetry remained unclear.

In the present paper we establish the explicit form of this relation through an appropriate extension of methods applied by CWZ (1969) in their analysis of the relation between linear and nonlinear realisations of Bose symmetries. The paper is organised as follows. In § 2 we show that an arbitrary superfield can always be represented in a 'splitting' form in which its components transform according to the nonlinear law (3). We formulate general prescriptions for passing to the splitting representation. In § 3, we give a complete solution to the related problem of constructing linearly transforming superfields from the nonlinear realisation quantities $\lambda(x)$, $\sigma_k(x)$. The developed methods are applicable, with obvious modifications, to any supersymmetry realised in some superspace. We compare our results with general theorems derived by CWZ (1969) in standard symmetries and find close correspondence between them.

2. The splitting representation of superfields

2.1. The general theory of nonlinear realisations of internal symmetries tells us that any linear multiplet of a given group can be converted into the direct sum of nonlinearity transforming fields by means of the group transformation with the Goldstone field as a parameter (CWZ 1969). The analogous theorem turns out to hold also in the case of supersymmetry.

Perform, in some superfield $\Phi_k(x, \theta)$, the local supertranslation with parameter $-\lambda(x)$:

$$\Phi_k(x, \theta) \to \Phi_k^\sigma(x, \theta) = \Phi_k\left(x - \frac{1}{2\mathrm{i}}\bar{\lambda}(x)\gamma\theta, \theta + \lambda(x)\right) \tag{4}$$

and examine how the shifted superfield $\Phi_k^\sigma(x, \theta)$ behaves under global supertransformations (1), (2). As $\Phi_k^\sigma(x, \theta)$ is a composite superfunction its infinitesimal variation consists of two pieces. First of them is the usual supertranslation according to rule (1) but at points $x'_\mu = x_\mu - (1/2\mathrm{i})\bar{\lambda}(x)\gamma_\mu\theta$, $\theta' = \theta + \lambda(x)$,

$$\delta_1\Phi_k^\sigma(x, \theta) = -\bar{\epsilon}\left[\frac{\partial}{2}(\gamma_\mu\theta')\frac{\partial}{1}(\gamma_\mu\theta')\frac{\partial}{\partial x'_\mu}\right]\Phi_k(x', \theta') \tag{5}$$

and the second is induced by the change of field $\lambda(x)$ in arguments x'_μ, θ',

$$\delta_2\Phi_k^\sigma(x, \theta) = -\frac{1}{2\mathrm{i}}\delta\bar{\lambda}(x)\gamma_\mu\theta\frac{\partial}{\partial x'_\mu}\Phi_k(x', \theta') + \delta\bar{\lambda}(x)\frac{\partial}{\partial\bar{\theta}'}\Phi_k(x', \theta') \tag{6}$$

where $\delta\lambda(x)$ is given by equation (2). Summing separate variations (5), (6) and taking into account the relation

$$\frac{\partial}{\partial x_\mu} = \left(\delta^\mu_\rho - \frac{1}{2\mathrm{i}}\partial^\mu\bar{\lambda}(x)\gamma_\rho\theta\right)\frac{\partial}{\partial x'_\rho} + \partial^\mu\bar{\lambda}(x)\frac{\partial}{\partial\bar{\theta}'}$$

we find that components of $\phi_k^\sigma(x, \theta)$ transform independently of each other, according to the nonlinear law (3),

$$\delta\Phi_k^\sigma(x,\theta)=\delta_1\Phi_k^\sigma(x,\theta)+\delta_2\Phi_k^\sigma(x,\theta)=\frac{1}{2\mathrm{i}}\bar{\epsilon}\gamma_\mu\lambda(x)\frac{\partial}{\partial x_\mu}\Phi_k^\sigma(x,\theta). \tag{7}$$

Thus, by changing variables any superfield with the linear transformation rule (1) can always be brought into the splitting form (4) in which it is represented by a set of nonlinearly transforming components.

As follows from equation (4) components of $\Phi_k^\sigma(x,\theta)$ are finite polynomials in anticommuting spinors $\lambda(x)$ and are linear in components of initial superfield $\Phi_k(x,\theta)$. They all can be represented as a result of repeated application of the operator

$$\mathscr{D}(\lambda)=\frac{\partial}{\partial\bar{\lambda}}+\frac{1}{2\mathrm{i}}(\gamma_\mu\lambda)\frac{\partial}{\partial x_\mu}$$

(which is nothing but the Salam–Strathdee (1975) spinor covariant derivative taken at $\theta=\lambda$) to the λ-polynomial $\Phi_k(x,\lambda(x))$:

$$\begin{aligned}
&A_k^\sigma(x)=\Phi_k(x,\lambda(x))=A_k(x)+\dots\\
&\psi_k^\sigma(x)=\mathscr{D}(\lambda)\Phi_k(x,\lambda(x))=\psi_k(x)+\dots\\
&\dots\\
&\mathrm{D}_k^\sigma(x)=\tfrac{1}{2}[\bar{\mathscr{D}}(\lambda)\mathscr{D}(\lambda)]^2\Phi_k(x,\lambda(x))=\mathrm{D}(x)+\dots,
\end{aligned}\tag{8}$$

the derivative $\partial/\partial x_\mu$ acting only on the first argument of $\Phi_k(x,\lambda)$ i.e. on linearly transforming components. It should be emphasised that the supersymmetric mapping (8) in contrast to the case of usual symmetries, with necessity includes field derivatives.

To avoid a possible misunderstanding, we point out that the existence of the splitting representation (4) for a superfield does not mean, in itself, that the superfield approach is equivalent to the nonlinear realisation, as far as $\lambda(x)$ remains an extra field unrelated to linearly transforming components. One-to-one correspondence between both approaches can be established only within the framework of spontaneously broken supersymmetry when superfields themselves involve the inhomogeneously transforming Goldstone component. In this case, it becomes possible to express $\lambda(x)$ through linear components by an equivalence transformation (see our subsequent paper).

2.2. Now let us study in more detail the structure of transformation (4) for some special superfields often encountered in practice.

Consider first the vector and spinor covariant derivatives

$$\frac{\partial}{\partial x_\mu}\Phi_k(x,\theta),\ \mathscr{D}_\alpha\Phi_k(x,\theta)=\left[\frac{\partial}{\partial\bar{\theta}^\alpha}+\frac{1}{2\mathrm{i}}(\gamma_\mu\theta)_\alpha\frac{\partial}{\partial x_\mu}\right]\Phi_k(x,\theta).$$

They are transformed as

$$\begin{aligned}
&\frac{\partial}{\partial x_\mu}\Phi_k(x,\theta)\to\Delta^\mu\Phi_k^\sigma(x,\theta)\\
&\mathscr{D}_\alpha\Phi_k(x,\theta)\to\Delta_\alpha\Phi_k^\sigma(x,\theta)
\end{aligned}\tag{9}$$

$$\begin{aligned}
&\Delta^\mu=(M^{-1})^\mu_\lambda\left[\nabla^\lambda-\nabla^\lambda\bar{\lambda}(x)\frac{\partial}{\partial\bar{\theta}}\right]\\
&\Delta_\alpha=\frac{\partial}{\partial\bar{\theta}^\alpha}+\frac{1}{2\mathrm{i}}(\gamma_\mu\theta)_\alpha\Delta^\mu
\end{aligned}\tag{10}$$

where the symbol ∇^ρ stands for the nonlinear realisation covariant derivative (Volkov and Akulov 1972, 1973, Pashnev 1974):

$$\nabla^\rho = (T^{-1})^\rho_\mu \frac{\partial}{\partial x_\mu}, \qquad T^\rho_\mu = \delta^\rho_\mu - \frac{1}{2\mathrm{i}} \partial^\rho \bar{\lambda}(x)\gamma_\mu \lambda(x) \tag{11}$$

and the θ-dependent matrix M^ρ_μ is defined by

$$M^\rho_\mu = \delta^\rho_\mu - \frac{1}{2\mathrm{i}} \nabla^\rho \bar{\lambda}(x)\gamma_\mu \theta. \tag{12}$$

Note that $\nabla^\rho\lambda(x)$, $\nabla^\rho\sigma_k(x)$ transform under (2) and (3) as fields $\sigma_k(x)$ themselves. Being formed only of covariant quantities, $\Delta^\mu\phi^\sigma_k(x, \theta)$, $\Delta_\alpha\phi^\sigma_k(x, \theta)$ are manifestly covariant with respect to transformations (2) and (7).

One more important class of superfields is given by chiral ones $\phi^\mp_k(x, \theta)$ defined as (Salam and Strathdee 1975)

$$\phi^\pm_k(x, \theta) = \exp(\mp\tfrac{1}{4}\bar{\theta}\, \not\partial \gamma_5 \theta) S^\pm_k(x, \theta_\pm) \tag{13}$$

where $S^\pm_k(x, \theta_\pm)$ depends only on two-component Grassmann spinors

$$\theta_\pm = \tfrac{1}{2}(1 \pm \mathrm{i}\gamma_5)\theta.$$

When $\phi^\pm_k(x, \theta)$ undergo supertransformation (1), the truncated superfields $S^\pm_k(x, \theta_\pm)$ transform as

$$S'^\pm_k(x, \theta_\pm) = S^\pm_k(x - \mathrm{i}\bar{\epsilon}\gamma\theta \mp \tfrac{1}{4}\bar{\epsilon}\gamma\gamma_5\epsilon, \theta_\pm - \epsilon_\pm) \tag{14}$$

thus realising by themselves representations of the same superalgebra. The splitting form of $S^\pm_k(x, \theta_\pm)$ may be arrived at again by means of the substitution $\epsilon \to -\lambda(x)$ in law (14):

$$S^\pm_k(x, \theta_\pm) \to S^{\sigma\pm}_k(x, \theta_\pm) = S^\pm_k(x + \mathrm{i}\bar{\lambda}(x)\gamma\theta_\pm \mp \tfrac{1}{4}\bar{\lambda}(x)\gamma\gamma_5\lambda(x), \theta_\pm + \lambda_\pm(x)). \tag{15}$$

As in deriving equation (7), one can readily check that components of $S^{\sigma\pm}_k(x, \theta_\pm)$ transform according to law (3). Note that, whereas $\phi^\pm_k(x, \theta)$ and $S^\pm_k(x, \theta_\pm)$ are connected by simple equations (13), the relation between splitting superfields $\phi^{\sigma\pm}_k(x, \theta)$ and $S^{\sigma\pm}_k(x, \theta_\pm)$ is essentially more complicated

$$\phi^{\sigma\pm}_k(x, \theta) = [1 \mp \tfrac{1}{4}\bar{\theta}\gamma_\mu\gamma_5\theta\mathscr{D}^\mu_\pm - \tfrac{1}{32}(\bar{\theta}\theta)^2\mathscr{D}^\mu_\pm\mathscr{D}_{\mu\pm}]S^{\sigma\pm}_k(x, \theta_\pm) \tag{16}$$

where

$$\mathscr{D}^\mu_\pm = (M^{-1}_\pm)^\mu_\rho\nabla^\rho - (M^{-1}_\pm)^\mu_\rho\nabla^\rho\lambda_\pm(x)\frac{\partial}{\partial\theta_\pm} \tag{17}$$

$$M_{\pm\rho}^{\ \mu} = \delta^\mu_\rho + \mathrm{i}\nabla^\mu\bar{\lambda}(x)\gamma_\rho\theta_\pm. \tag{18}$$

3. Superfields as functions of the nonlinear realisation fields

3.1. Once general formulae for transition from the linear realisation to the nonlinear one are established, we are in a position to answer the question of the possibility of constructing linearly transforming superfields out of objects $\lambda(x)$, $\sigma_k(x)$.

Relation (4) may be inverted to give

$$\phi_k(x, \theta) = \phi^\sigma_k[\tilde{X}(x, \theta), \theta - \lambda(\tilde{X}(x, \theta))] \tag{19}$$

where $\tilde{X}_\mu(x, \theta)$ obeys the functional equation

$$\tilde{X}_\mu(x, \theta) - \frac{1}{2\mathrm{i}}\bar{\lambda}(\tilde{X})\gamma_\mu\theta = x_\mu. \tag{20}$$

Iterating equation (20), one may represent $\tilde{X}_\mu(x, \theta)$ as a finite polynomial in nilpotent quantities θ, $\lambda(x)$ and derivatives of $\lambda(x)$,

$$\tilde{X}_\mu(x, \theta) = x_\mu - \frac{1}{2\mathrm{i}}\bar{\theta}\gamma_\mu\lambda(x) + \frac{1}{4}(\bar{\theta}\gamma_\rho\lambda(x))(\partial^\rho\bar{\lambda}(x)\gamma_\mu\theta) + \dots.$$

So relation (19) provides the decomposition of $\phi_k(x, \theta)$ in terms of the nonlinear realisation quantities. It can be regarded as a supersymmetric counterpart of the polar decomposition of multiplets in internal symmetries, the Goldstone spinor $\lambda(x)$ being an analogue of angular variables and components of $\phi_k^\sigma(x, \theta)$ playing the role of radial variables.

The right-hand side of equation (19) is the superposition of the following θ-polynomials

$$\tilde{\lambda}_\alpha(x, \theta) = \lambda_\alpha[\tilde{X}(x, \theta)] - \theta_\alpha \tag{21}$$

$$\tilde{\sigma}_k(x, \theta) = \sigma_k[\tilde{X}(x, \theta)] \tag{22}$$

which admit a simple interpretation: they may be viewed as generated from fields $\lambda_\alpha(x)$, $\sigma_k(x)$ by finite supertransformations with parameter $-\theta_\alpha$. It is proved in appendix A that polynomials (21) and (22) transform under (2) and (3) according to the infinitesimal form of linear law (1) and hence possess in themselves the properties of superfields (the proof is analogous to the derivation of equation (7)).

So, given nonlinearly transforming fields $\lambda_\alpha(x)$, $\sigma_k(x)$, by substitution $x_\mu \to \tilde{X}_\mu(x, \theta)$ in them one can construct linearly transforming superfields with an arbitrary external spin.

The components of basis superfields (21) and (22) are evaluated in terms of $\lambda_\alpha(x)$, $\sigma_k(x)$ and their derivatives by the simple iteration formula:

$$\frac{\partial}{\partial\theta_\alpha}\begin{pmatrix}\tilde{\lambda}_\beta(x, \theta)\\ \tilde{\sigma}_k(x, \theta)\end{pmatrix} = \begin{pmatrix}-\delta^\alpha_\beta\\ 0\end{pmatrix} - \frac{1}{2\mathrm{i}}(\bar{\lambda}(\tilde{X})\gamma_\mu)^\alpha\frac{\partial}{\partial x_\mu}\begin{pmatrix}\lambda_\beta(\tilde{X})\\ \sigma_k(\tilde{X})\end{pmatrix} \tag{23}$$

which is derived by using equation (20). For instance, the first two components of superfield (21) are given by

$$\tilde{A}_\alpha(x) = \lambda_\alpha(x)$$

$$\tilde{\psi}_{\alpha\beta}(x) = C_{\beta\alpha} - \frac{1}{2\mathrm{i}}(\gamma_\mu\lambda(x))_\beta\,\partial^\mu\lambda_\alpha(x)$$

$C_{\beta\alpha}$ being the charge conjugation matrix. One directly checks that $\tilde{A}_\alpha(x)$ transforms really as the first component of superfield,

$$\delta\tilde{A}_\alpha(x) = -\bar{\varepsilon}\tilde{\psi}_\alpha(x).$$

By the way, equation (23) implies that the covariant spinor derivative $\mathscr{D}_\alpha$ reduces on superfields (21) and (22) to usual derivative $\partial/\partial x_\mu$,

$$\mathscr{D}_\alpha\begin{pmatrix}\tilde{\lambda}_\beta(x, \theta)\\ \tilde{\sigma}_k(x, \theta)\end{pmatrix} = \begin{pmatrix}C_{\alpha\beta}\\ 0\end{pmatrix} - \frac{1}{2\mathrm{i}}(\gamma_\mu\tilde{\lambda}(x, \theta))_\alpha\frac{\partial}{\partial x_\mu}\begin{pmatrix}\tilde{\lambda}_\beta(x, \theta)\\ \tilde{\sigma}_k(x, \theta)\end{pmatrix}.$$

The reason is that variables x_μ, θ_α enter into quantities (21) and (22) not independently but through the fixed combination $\tilde{X}_\mu(x, \theta)$.

3.2. Having the basis set of superfields (21) and (22) we may construct new superfields either by multiplying the basis ones and acting on thus obtained products by usual and spinor derivatives or applying the algorithm $x_\mu \to \tilde{X}_\mu(x, \theta)$ directly to the nonlinear realisation covariant derivatives $\nabla_\rho \lambda(x)$, $\nabla_\mu \sigma_k(x)$. In such a way a superfield with any given external spin can be attained. Irreducible superfields with definite superspins are extracted, as usual, with the help of projection operators (Salam and Strathdee 1975, Sokatchev 1975). In particular by such a procedure one may construct chiral superfields (13) which are known to carry superspins equal to values j of corresponding external spins (j or $-j$ depending on chirality). However, there exists a more direct and elegant recipe to build chiral superfields out of the nonlinear realization fields. Namely, one should start from the following set of 'truncated' θ-polynomials:

$$\tilde{\lambda}_\pm(x, \theta_\pm) = \lambda_\pm(\tilde{X}^\pm) - \theta_\pm \tag{24}$$

$$\tilde{\sigma}_k^\pm(x, \theta_\pm) = \sigma_k(\tilde{X}^\pm) \tag{25}$$

where θ-functions $\tilde{X}_\mu^\pm(x, \theta_\pm)$ satisfy the equations

$$\tilde{X}_\mu^\pm(x, \theta_\pm) + \mathrm{i}\bar{\lambda}^c(\tilde{X}^\pm)\gamma_\mu\theta_\pm \pm \tfrac{1}{4}\bar{\lambda}^c(\tilde{X}^\pm)\gamma_\mu\gamma_5\lambda(\tilde{X}^\pm) = x_\mu \tag{26}$$

and

$$\bar{\lambda}^c(\tilde{X}^\pm) \equiv c^{-1}\lambda(\tilde{X}^\pm).$$

This set emerges as a result of inverting equation (15). In the same fashion as it is done in appendix A for superfields (21) and (22) we may be convinced that objects (24) and (25) transform just as required by the linear law (14) and therefore form a proper basis for constructing chiral superfields. The explicit form of simplest scalar multiplets made of $\lambda(x)$ and its derivatives by multiplying basis polynomials (24) is given in appendix B.

Note that Zumino (1974) found the nonlinear realisation different from (2) and (3), with the Goldstone spinor $\chi(x)$ transforming as follows†:

$$\delta\chi(x) = \epsilon + \frac{1}{2\mathrm{i}}\bar{\epsilon}\gamma_\mu\chi(x)\,\partial^\mu\chi(x) + \frac{1}{2\mathrm{i}}\bar{\epsilon}\gamma_5\gamma_\mu\chi(x)\gamma_5\,\partial^\mu\chi(x) \tag{27}$$

and mentioned that he had succeeded in constructing scalar multiplets just of spinors $\chi(x)$. Our remark is that realisation (27) is related to the canonical one (2) through the equivalence redefinition of field $\chi(x)$ just as, for instance, different nonlinear realisations for pions are related (see e.g. Weinberg 1968). Therefore any linearly transforming function of $\chi(x)$ can be constructed of $\lambda(x)$ and its derivatives using one of the general methods described above. The connection between spinors $\lambda(x)$ and $\chi(x)$ is given by

$$\chi(x) = \tfrac{1}{2}(1 + \mathrm{i}\gamma_5)\lambda(Z^-(x, \lambda)) + \tfrac{1}{2}(1 - \mathrm{i}\gamma_5)\lambda(Z^+(x, \lambda)) \tag{28}$$

where $Z_\mu^\pm$ are determined from the functional equations:

$$Z_\mu^\pm(x, \lambda) \pm \tfrac{1}{4}\bar{\lambda}^c(Z^\pm)\gamma_\mu\gamma_5\lambda(Z^\pm) = x_\mu. \tag{29}$$

† An interesting property of realisation (27) is its reducibility, in the sense that left- and right-handed parts of $\chi(x)$ transform independently of each other

$$\delta\chi_\pm = \epsilon_\pm - i\bar{\epsilon}\gamma_\mu\chi_\pm\,\partial^\mu\chi_\pm.$$

(In contrast to formulae of equivalence redefinition for pions relation (28) includes derivatives of field $\lambda(x)$.) By methods analogous to those used in appendix A one may verify that under transformation (2) $\chi(x)$ defined as in (28) transforms as required by the law (27). The Majorana nature of $\chi(x)$ is readily established with taking into account the complex conjugation rule

$$(Z_\mu^\pm)^\dagger = Z_\mu^\mp$$

which follows from the definition of $Z_\mu^\pm(x, \lambda(x))$.

3.3. In the case of usual symmetries the procedure of composing linear representations in terms of the nonlinear realisation is governed by a number of general theorems derived by CWZ (1969). It turns out that all these theorems admit the supersymmetric extension.

One of them determines which representations of a given group can be constructed starting with a set of σ-fields belonging to a certain irreducible representation R of the stability subgroup. Namely, it states that it is possible to build only those representations which contain R when restricted to the stability subgroup. If R enters into some representation m times there exist m independent ways to construct the latter. To verify that analogous statements are valid also in the case of supersymmetry one has to examine more carefully general relation (19). Digressing from the manner in which equation (19) has been obtained one observes that any Lorentz multiplet of σ-fields in the right-hand side of equation (19) can be put zero without spoiling the supertransformation properties of $\phi_k(x, \theta)$ in the left-hand side (such conditions, being manifestly covariant, produce covariant relations between components of $\phi_k(x, \theta)$ analogous to the relation $\sigma^2 + \pi^2 = 1$ in chiral dynamics†). Thus superfield $\phi_k(x, \theta)$ can be restored (nonlinearly) by making use of at least one set of σ-fields which transforms under the Lorentz group like one of those irreducible Lorentz multiplets to which the components of $\phi_k(x, \theta)$ belong. Clearly if this multiplet is encountered among components of $\phi_k(x, \theta)$ several times, so many non-equivalent superfields $\phi_k(x, \theta)$ built with the help of the same set of σ-fields exist. For example, given the Lorentz scalar $\sigma(x)$, one may form three general scalar superfields

$$\phi^1(x, \theta) = \tilde{\sigma}(x, \theta), \qquad \phi^2(x, \theta) = (\bar{\tilde{\lambda}}(x, \theta)\tilde{\lambda}(x, \theta))\tilde{\sigma}(x, \theta),$$
$$\phi^3(x, \theta) = (\bar{\tilde{\lambda}}(x, \theta)\tilde{\lambda}(x, \theta))^2\tilde{\sigma}(x, \theta)$$

corresponding to a number of scalar components in such a superfield. Starting with the pseudoscalar field $\sigma_5(x)$ it is possible to construct only one scalar superfield

$$\phi(x, \theta) = \bar{\tilde{\lambda}}(x, \theta)\gamma_5\tilde{\lambda}(x, \theta)\sigma_5(\tilde{X})$$

again in agreement with the aforementioned theorem of CWZ because the stability subgroup of the nonlinear realisation in question includes, along with the proper Lorentz transformations, also the space–time reflections‡. It is perhaps worth noting

† The detailed discussion of similar conditions for the case of an arbitrary internal symmetry is given by Ivanov (1976).

‡ Remember that the Poincaré translations belong to the quotient space, usual coordinate x_μ being regarded as the Goldstonion associated with the generator P_μ (Volkov and Akulov 1972, 1973).

once more that the role of σ-fields can be played by various products of covariant derivatives $\nabla_\rho\lambda(x)$, $\nabla_\rho\sigma_k(x)$.

We readily recover also the remaining theorems of CWZ. As an example, take the important theorem which says that of the Goldstone fields themselves (without using their derivatives) it is possible to make up only those representations of a given group which contain invariants of the stability subgroup (in fact it is a corollary of the theorem already discussed). In the supersymmetry case, starting with $\lambda(x)$ alone, we are able to form only scalar, pseudoscalar, spinor and pseudovector superfields (multiplying basis spinor superfields (21) and taking into account their Grassmann and Majorana nature.) But only these superfields include the Lorentz scalar components. Thus the validity of the theorem we are discussing is clear. The specific feature of the supersymmetry case is that superfields thus constructed although beginning with terms without derivatives, inevitably include derivatives of $\lambda(x)$ in subsequent terms. Of course, performing the replacement $x_\mu \to \tilde{X}_\mu(x, \theta)$ in the covariant derivatives $\nabla_\rho\lambda(x)$ one may obtain superfields of higher external spins but all those will begin with derivatives of $\lambda(x)$.

3.4. It should be noticed that results of §§ 2 and 3 can straightforwardly be extended to supersymmetries more complicated than the standard one we dealt with here. Indeed, in order to define a mapping of the type given by equations (4) and (19) we need only to know how a supersymmetry is realised in a relevant superspace. Once this realisation is found, the splitting form of an arbitrary superfield is arrived at by the substitution of the corresponding nonlinear realisation Goldstone fermion (taken with sign minus) for a spinor parameter into the supertransformed superfield. Inverting obtained relations one deduces algorithms for constructing linear representations of a given supergroup in terms of objects of relevant nonlinear realisation. It would be interesting to relate, in such a manner, linear (Keck 1975) and nonlinear (Zumino 1977) realisations of the graded group OSp(1, 4) which seems to play the important role in supergravity (MacDowell and Mansouri 1977, Deser and Zumino 1977).

4. Conclusion

In this paper we focused on formal, purely group theory aspects of the relationship between the superfield approach and the Volkov–Akulov nonlinear realisation. Its manifestations at the level of Lagrangians with spontaneously broken supersymmetry and related questions will be discussed in a subsequent paper.

We note finally that algorithms and formulae presented here may appear to be useful for constructing the superspace formulation of the spontaneously broken supergravity and, in particular, for transforming into the superfield language the supersymmetric Higgs effect which was treated up to now within the nonlinear realisation formalism (Volkov and Soroka 1973, Deser and Zumino 1977).

Acknowledgments

We express our deep gratidude to Professor V I Ogievetsky for interest in the work and constructive remarks. We thank Drs E Sokatchev and B Zupnik for useful discussions.

Appendix A

We prove here that the objects $\tilde{\sigma}_k(x, \theta)$ and $\tilde{\lambda}_\alpha(x, \theta)$ defined by equations (21) and (22) undergo linear supertransformation when their constituents $\lambda_\alpha(x)$, $\sigma_k(x)$ transform according to the nonlinear laws (2) and (3). The proof is the same for $\tilde{\sigma}_k(x, \theta)$ and $\tilde{\lambda}_\alpha(x, \theta)$ therefore it is sufficient to perform it, say for $\tilde{\lambda}_\alpha(x, \theta)$.

The rule of varying $\tilde{\lambda}_\alpha(x, \theta)$ is as follows. One has to vary function $\lambda_\alpha(\tilde{X}(x, \theta))$ at fixed argument $\tilde{X}_\mu$ by law (2) and, besides, to take into account the effect of changing $\tilde{X}_\mu(x, \theta)$ due to the nonlinear shift of field $\lambda(x)$, variables x_μ, θ_α remaining unaffected. So the supervariation of $\tilde{\lambda}_\alpha(x, \theta)$ is

$$\delta\tilde{\lambda}_\alpha(x, \theta) = \epsilon_\alpha + \frac{1}{2i}(\bar{\epsilon}\gamma_\mu\lambda(\tilde{X}))\frac{\partial}{\partial\tilde{X}_\mu}\lambda_\alpha(\tilde{X}) + \delta\tilde{X}_\mu\frac{\partial}{\partial\tilde{X}_\mu}\lambda_\alpha(\tilde{X}) \tag{A.1}$$

where $\delta\tilde{X}_\mu$ satisfies the equation

$$\left(\delta^\rho_\mu - \frac{1}{2i}\frac{\partial}{\partial\tilde{X}_\rho}\bar{\lambda}(\tilde{X})\gamma_\mu\theta\right)\delta\tilde{X}_\rho = \frac{1}{2i}\bar{\epsilon}\gamma_\mu\theta + \frac{1}{4}(\bar{\lambda}(\tilde{X})\gamma_\rho\epsilon)\frac{\partial}{\partial\tilde{X}_\rho}\bar{\lambda}(\tilde{X})\gamma_\mu\theta \tag{A.2}$$

which results from varying equation (20). Observing that

$$\frac{\partial}{\partial\tilde{X}_\mu} = \frac{\partial x_\rho}{\partial\tilde{X}_\mu}\frac{\partial}{\partial x_\rho} = \left(\delta^\mu_\rho - \frac{1}{2i}\frac{\partial}{\partial\tilde{X}_\mu}\bar{\lambda}(\tilde{X})\gamma_\rho\theta\right)\frac{\partial}{\partial x_\rho} \tag{A.3}$$

and inserting (A.2), (A.3) into (A.1) one finds

$$\delta\tilde{\lambda}_\alpha(x, \theta) = \epsilon_\alpha + \frac{1}{2i}\bar{\epsilon}\gamma_\mu\lambda(\tilde{X})\frac{\partial}{\partial x_\mu}\lambda_\alpha(\tilde{X}) + \frac{1}{2i}\bar{\epsilon}\gamma_\mu\theta\frac{\partial}{\partial x_\mu}\lambda_\alpha(\tilde{X}). \tag{A.4}$$

Using relation (23) it is easy to cast (A.4) into the form

$$\delta\tilde{\lambda}_\alpha(x, \theta) = -\bar{\epsilon}\left[\frac{\partial}{\partial\bar{\theta}} - \frac{1}{2i}(\gamma_\mu\theta)\frac{\partial}{\partial x_\mu}\right]\tilde{\lambda}_\alpha(x, \theta) \tag{A.5}$$

which is just the standard superfield transformation.

Appendix B

The simplest chiral 'truncated' superfields which one may compose of θ-polynomials (24) are scalar ones:

$$S^\pm(x, \theta_\pm) = \bar{\tilde{\lambda}}^c_\pm(x, \theta_\pm)\lambda_\pm(x, \theta_\pm) = \bar{\lambda}^c_\pm(\tilde{X}^\pm)\lambda_\pm(\tilde{X}^\pm) - 2\bar{\theta}_\pm\lambda_\pm(\tilde{X}^\pm) + \bar{\theta}_\pm\theta_\pm. \tag{B.1}$$

As an example, we write down explicitly the components of $S^+(x, \theta_+)$ in terms of $\lambda(x)$ and its derivatives. The most compact notation is achieved by using the spinor function

$$\omega(x) = \lambda(Z^+(x, \lambda))$$

where $Z^+_\mu(x, \lambda)$ is defined by one of equations (29). In terms of $\omega(x)$ we have

$$\begin{aligned} &\lambda(\tilde{X}^+) = \omega(\hat{X}^+) \\ &\hat{X}^+_\mu + i\bar{\omega}^c(\hat{X}^+)\gamma_\mu\theta_+ = x_\mu. \end{aligned} \tag{B.2}$$

Expanding both sides of (B.1) in powers of θ_+ and making use of equations (B.2) we find for components of $S^+(x, \theta_+)$

$$\begin{aligned}
A_+(x) &= \bar{\omega}^c(x)\tfrac{1}{2}(1+\mathrm{i}\gamma_5)\omega(x) \\
\psi_+(x) &= -(1+\mathrm{i}\gamma_5)\omega(x)+\tfrac{1}{2}\mathrm{i}(1+\mathrm{i}\gamma_5)(\gamma_\mu\omega)\,\partial^\mu[\bar{\omega}^c\tfrac{1}{2}(1+\mathrm{i}\gamma_5)\omega] \\
F_+(x) &= 2-\mathrm{i}\bar{\omega}^c\gamma_\mu\tfrac{1}{2}(1+\mathrm{i}\gamma_5)\,\partial^\mu\omega \\
&\quad +\tfrac{1}{2}\bar{\omega}^c\gamma_\rho\gamma_\mu\tfrac{1}{2}(1-\mathrm{i}\gamma_5)\,\partial^\rho\omega\,\partial^\mu[\bar{\omega}^c\tfrac{1}{2}(1+\mathrm{i}\gamma_5)\omega] \\
&\quad +\tfrac{1}{4}\bar{\omega}^c\tfrac{1}{2}(1-\mathrm{i}\gamma_5)\omega\Box[\bar{\omega}^c\tfrac{1}{2}(1+\mathrm{i}\gamma_5)\omega].
\end{aligned} \tag{B.3}$$

Exploiting the supertransformation properties of $\omega(x)$,

$$\delta\omega(x)=\epsilon+\frac{1}{2\mathrm{i}}\bar{\epsilon}\gamma_\mu(1-\mathrm{i}\gamma_5)\omega(x)\,\partial^\mu\omega(x),$$

we checked 'by hand' that functions (B.3) really form the left-handed scalar supermultiplet. Note that components of $S^-(x, \theta_-)$ are simply the complex conjugates of (B.3).

Note added in proof. The explicit formulae expressing components of a scalar supermultiplet as functions of the nonlinear realisation Goldstone fermion have also been obtained recently by M. Roček (1978) through direct and tedious calculations. They coincide, up to the equivalence redefinition of $\lambda(x)$, with ours (B.3), derived using the general procedure (for the choice $\alpha=\mathrm{i}$ in Roček's article, both sets of formulae are in fact identical with each other). We note that the basic ingredients of the approach developed in our present paper have already been briefly outlined in our preprint (1977).

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